FINITE TERMINATION GAMES WITH TIE*

BY

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ABSTRACT

We consider the class of finite two-person games with perfect information in which the last player who can make a legal move either wins or ends the game in a tie. We define an equivalence relation over this class and exhibit a complete set of representatives for the equivalence classes defined in terms of one-pile Nim games.

We shall consider in this paper a class of two-person games with perfect information in which the players play alternately, at each turn select a move from a finite set of alternatives, and complete a play of the game after finitely many moves. If we add the condition that the first player who cannot make a legal move loses, then every such game may be represented as follows.

DEFINITION 1. $G_0 = \emptyset$, $G_{k+1} = \{a : a \subseteq G_k\}$, and $G = \bigcup_{k \in \omega} G_k$.

To each $a \in G$ corresponds the game $\mathcal{G}_a$ in which player I must choose some $a_1 \in a$, player II some $a_2 \in a_1$, and so on until some player chooses $\emptyset$ and thus wins the game. If we think of a position in a game as being the set of all permissible succeeding positions, it is clear that any game of the type described is equivalent to some $\mathcal{G}_a$. The theory of such games is well-developed (cf. [1])*.

We shall enlarge the class of games considered to allow for the possibility of a tie. Now a player may be faced with a position from which he can make no legal move either because he has lost or because the game has ended in a tie. Such games can all be represented in the form $\mathcal{G}_a$ for $a \in H$ defined as follows.

DEFINITION 2. $H_0 = \{*\}$, $H_{k+1} = \{a : a \subseteq H_k\}$, and $H = \bigcup_{k \in \omega} H_k$.

In this paper we provide an analysis of such games. As motivation for interest

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in these games, we note that chess is a finite termination game with tie.

We first define recursively the *value* $|a|$ of any $a \in H$.

DEFINITION 3. $$|*| = 0, |\varnothing| = -1, \text{ and otherwise}$$
$$|a| = -\min\{|x| : x \in a\}.$$

Just as in the case of win-lose games, $|a| = +1$ iff player I has a winning strategy in $\mathcal{G}_a$ and $|a| = -1$ iff player II has a winning strategy. If $|a| = 0$, either player can force the game to end in a tie. We call attention to the following obvious facts which will be used repeatedly without reference.

LEMMA 4. *For any* $a \in H$, $a \neq *$:

(i) $|a| = +1 \leftrightarrow \exists x(x \in a. |x| = -1)$;

(ii) $|a| = 0 \leftrightarrow \forall x(x \in a \rightarrow |x| \geqq 0). \exists x(x \in a. |x| = 0)$;

(iii) $|a| = -1 \leftrightarrow \forall x(x \in a \rightarrow |x| = +1)$.

DEFINITION 5. For any $a, b \in H$:

(i) $a \oplus * = * \oplus b = *$; if $a \neq * \neq b$,
$$a \oplus b = \{x \oplus b : x \in a\} \cup \{a \oplus y : y \in b\};$$

(ii) $a \sim b \leftrightarrow \forall c \in H(|a \oplus c| = |b \oplus c|)$.

Of course it is immediate that H is closed under $\oplus$. The game $\mathcal{G}_{a \oplus b}$ is played by playing $\mathcal{G}_a$ and $\mathcal{G}_b$ simultaneously with each player at his turn moving in one game or the other. The following properties of $\oplus$ and $\sim$ are easily derived from the definitions.

LEMMA 6. (i) $\oplus$ *is commutative and associative*;

(ii) $\sim$ *is an equivalence relation*;

(iii) $a \oplus \varnothing = a$;

(iv) $a \sim b \rightarrow |a| = |b|$;

(v) $a \sim b \rightarrow a \oplus c \sim b \oplus c$.

In the analysis of win-lose games, the Nim games play an important role. With the usual set-theoretic representation of a natural number m as $\{0, 1, \cdots, m-1\}$, $m \in G$ and $\mathcal{G}_m$ is one-pile Nim with m stones: each player may remove any number of stones from the pile. More interesting are games $\mathcal{G}_{m_1 \oplus \cdots \oplus m_k}$ which start with k piles having, respectively, $m_1, \cdots, m_k$ stones. The main result of [1] is that every $a \in G$ is equivalent ($\sim_G$ defined by $\forall c \in G \cdots$) to a unique natural number n. There is even a simple process known as *Nim addition* for computing n such that

$m_1 \oplus m_2 \sim n$—one writes m_1 and m_2 in binary notation and adds each column modulo 2. Our aim here is to provide a similar set of representatives for the equivalence classes of H.

LEMMA 7. *For any natural number* m, $|m \oplus m| = -1$.

PROOF. For $m = 0$ this is immediate. For $m > 0$,

$$m \oplus m = \{m \oplus n : n < m\}.$$

For each $n < m$, $n \oplus n \in m \oplus n$ and $|n \oplus n| = -1$ by the induction hypothesis, so $|m \oplus n| = +1$. Hence $|m \oplus m| = -1$.

COROLLARY 8. *For any natural numbers m and n:*
(i) $m \neq n \to |m \oplus n| = +1$;
(ii) $m \sim n \to m = n$.

LEMMA 9. *For any $a, b \in H$ and any natural number m:*

(i) $a \sim m \to |a \oplus m| = -1$;
(ii) $* \in a \to |a \oplus b| \geqq 0$;
(iii) $m \in a \to |a \oplus m| = +1$;
(iv) $a \subseteq \omega,\ m \notin a \to |a \cup \{*\} \oplus m| = 0$.

PROOF. (i) If $a \sim m$, then $|a \oplus m| = |m \oplus m| = -1$. (ii) If $* \in a$, then $* \oplus b \in a \oplus b$. Since $|* \oplus b| = 0$, $\min\{|x| : x \in a \oplus b\} \leqq 0$ so $|a \oplus b| \geqq 0$. (iii) is immediate from Lemma 7. (iv) Assume that $a \subseteq \omega$ and $m \notin a$. Any element c of $a \cup \{*\} \oplus m$ is of one of the following forms:
$c = * \oplus m$: then $|c| = 0$ by definition;
$c = n \oplus m$ for $n \in a$: then $|c| = +1$ by 8(i);
$c = a \cup \{*\} \oplus n$ for $n < m$: then $|c| \geqq 0$ by (ii).
Hence the conditions of 4(ii) are satisfied so $|a \cup \{*\} \oplus m| = 0$.

DEFINITION 10. $\bar{H} = \omega \cup \{c \cup \{*\} : c \subseteq \omega,\ c \text{ finite}\}$.

Thus $\bar{H} \subseteq H$ and we aim to show that every $a \in H$ is equivalent to a unique $\bar{a} \in \bar{H}$. We show uniqueness first.

LEMMA 11. *For any $a, b \in \bar{H}$, if $a \sim b$, then $a = b$.*

PROOF Suppose $a, b \in \bar{H}$ and $a \sim b$. If $a, b \in \omega$, it follows from 8 (ii) that $a = b$. If $a = c \cup \{*\}$ and $b \in \omega$, then by 9(i), $|a \oplus b| = -1$, but by 9(ii), $|a \oplus b| \geqq 0$, a contradiction. The remaining case is $a = c \cup \{*\}$ and $b = d \cup \{*\}$. If $a \neq b$, there exists m such that (say) $m \in a$ but $m \notin b$. Then by 9(iii), $|a \oplus m| = +1$ but by 9(iv), $|b \oplus m| = 0$. Hence $a \nsim b$, which contradicts the hypothesis.

Let $\rho(a)$ be the smallest k such that $a \in H_k$. Note that $x \in a$ implies $\rho(x) < \rho(a)$.

LEMMA 12. *For any $a, b \in H$:*

(i) $|a| = +1,\ |b| = 0 \to |a \oplus b| \geqq 0$;

(ii) $|a| = -1 \to |a \oplus b| = |b|$.

PROOF. We prove (i) and (ii) simultaneously by induction on $\rho(a) + \rho(b)$. Both are vacuous for $a = *$ and obvious for $b = *$. Hence we assume $\rho(a) + \rho(b) > 0$. For (i) suppose $|a| = +1$ and $|b| = 0$. Then for some $x \in a$, $|x| = -1$ and $\rho(x) + \rho(b) < \rho(a) + \rho(b)$, so by (ii) of the induction hypothesis, $|x \oplus b| = |b| = 0$. Hence $\min\{|c| : c \in a \oplus b\} \leqq 0$ so $|a \oplus b| \geqq 0$. For (ii) suppose $|a| = -1$. There are three cases:

Case 1. $|b| = +1$. There exists $y \in b$ with $|y| = -1$. By the induction hypothesis (ii) $|a \oplus y| = |y| = -1$. Since $a \oplus y \in a \oplus b$, $|a \oplus b| = +1 = |b|$.

Case 2. $|b| = 0$. For all $y \in b$, $|y| \geqq 0$, so by (ii) of the induction hypothesis, $|a \oplus y| \geqq 0$. There is at least one $y \in b$ with $|y| = 0 = |a \oplus y|$. Since $|a| = -1$, all $x \in a$ have $|x| = +1$ so by (i) of the induction hypothesis, $|x \oplus b| \geqq 0$. Hence $\min\{|c| : c \in a \oplus b\} = 0$, and $|a \oplus b| = 0 = |b|$.

Case 3. $|b| = -1$. For all $y \in b$, $|y| = +1$, so by (ii) of the induction hypothesis, $|a \oplus y| = |y| = +1$. Likewise for all $x \in a$, $|x \oplus b| = |x| = +1$ Hence $\min\{|c| : c \in a \oplus b\} = +1$, and $|a \oplus b| = -1 = |b|$.

COROLLARY 13. *For any $a \in H$ and any natural number m:*

(i) $|a| = -1 \to a \sim \varnothing$,

(ii) $|a \oplus m| = -1 \to a \sim m$.

PROOF. (i) By 12 (ii) if $|a| = -1$, then for any b, $|a \oplus b| = |b| = |\varnothing \oplus b|$. (ii) Assume $|a \oplus m| = -1$ so by (i), $a \oplus m \sim \varnothing$. By Lemma 6, $a \oplus m \oplus m \sim \varnothing \oplus m = m$. But by Lemma 7, $|m \oplus m| = -1$ so by (i) again, $m \oplus m \sim \varnothing$. Hence $a \oplus m \oplus m \sim a \oplus \varnothing = a$. By transitivity of $\sim$, $a \sim m$.

LEMMA 14. *For any $a, b \in H$, if $|a \oplus b| = -1$, then for some natural number m, $a \sim m$ and $b \sim m$.*

PROOF. We prove the lemma by induction on $\rho(a) + \rho(b)$. If $|a \oplus b| = -1$, we have in particular that for all $x \in a$, $|x \oplus b| = +1$. Hence for all $x \in a$

$$\exists z \in x(|z \oplus b| = -1) \text{ or } \exists y \in b(|x \oplus y| = -1).$$

Case 1. $\exists x \in a \exists z \in x(|z \oplus b| = -1)$. Then by the induction hypothesis $b \sim m$

for some natural number m. Then $|a \oplus m| = |a \oplus b| = -1$ so by 13 (ii), $a \sim m$ also.

Case 2. $\forall x \in a \ \exists y \in b(|x \oplus y| = -1)$. By the induction hypothesis, for each $x \in a$ there exists a natural number m_x such that $x \sim m_x$. Corollary 8 implies that m_x is unique. Let

$$m = \text{least } p[p \notin \{m_x : x \in a\}].$$

We show first that $|a \oplus m| = -1$. For every $n < m$ there exists $x \in a$ such that $n = m_x$. By 9(i), $|x \oplus n| = |x \oplus m_x| = -1$. Hence $|a \oplus n| = +1$. On the other hand, for every $x \in a$, $m_x \neq m$ so by 8(i), $|x \oplus m| = |m_x \oplus m| = +1$. Thus for every $c \in a \oplus m$, $|c| = +1$ so $|a \oplus m| = -1$. Now by 13 (ii), $a \sim m$ and $|m \oplus b| = |a \oplus b| = -1$ so $b \sim m$ also.

THEOREM. *For every $a \in H$ there exists a unique $\bar{a} \in \bar{H}$ such that $a \sim \bar{a}$.*

PROOF. The uniqueness of $\bar{a}$ follows from Lemma 11. It is easy to prove by induction that for all b, $|\{*\} \oplus b| = 0$. Hence $* \sim \{*\}$ and we set $\bar{*} = \{*\}$. Suppose now that $\rho(a) > 0$ and $\bar{x}$ exists for all $x \in a$. If $a \sim m$ for some natural number m we are done, so assume otherwise. Set

$$\bar{a} = \{\bar{x} : x \in a, \ \bar{x} \in \omega\} \cup \{*\}.$$

Then $\bar{a} \in \bar{H}$ and it remains to show $a \sim \bar{a}$. Let b be an arbitrary element of H. It follows from Lemma 14 that $|a \oplus b| \geqq 0$ and from Lemma 9 (ii) that $|\bar{a} \oplus b| \geqq 0$. Hence it suffices to show that $|a \oplus b| = +1$ iff $|\bar{a} \oplus b| = +1$ We have

$$\begin{aligned} |\bar{a} \oplus b| = +1 &\leftrightarrow \exists z \in \bar{a}(|z \oplus b| = -1) \\ &\leftrightarrow \exists x \in a(\bar{x} \in \omega, \ |\bar{x} \oplus b| = -1) \\ &\leftrightarrow \exists x \in a(|x \oplus b| = -1) \\ &\leftrightarrow |a \oplus b| = +1. \end{aligned}$$

The first and fourth equvalences use again the fact that for any y, $|\bar{a} \oplus y| \geqq 0$ and $|a \oplus y| \geqq 0$. The second equivalence is just the definition of $\bar{a}$. Half of the third follows from $x \sim \bar{x}$. For the right-to-left direction, if $|x \oplus b| = -1$, then by Lemma 14, $x \sim m$ for some m. Since $m \in \bar{H}$, $m = \bar{x}$ by uniqueness.

Of course the proofs provide us with an algorithm for computing $\bar{a}$ from a. It would be interesting to know if there is a simple algorithm for computing $\overline{a \oplus b}$ for $a, b \in \bar{H}$ similar to Nim additon.

We have no conjectures as to how much further (if any) this method can be carried. An obvious candidate is the class of games obtained by adding a second atom with value $+1$.

REFERENCE

1. JOHN C. HOLLADAY, *Cartesian Products of Termination Games*, Contributions to the theory of games, Ed. M. Dresher, A.W. Tucker, and P. Wolfe, Annals of Math. Studies No. 39, Princeton University Press, Princeton, N.J., 1957, 189–200.

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